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Gravitational Anomalies in Noncommutative Field Theory

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Abstract

Gravitational axial and chiral anomalies in a noncommutative space are examined through the explicit perturbative computation of one-loop diagrams in various dimensions. The analysis depend on how gravity is coupled to noncommutative matter fields. Delbourgo-Salam computation of the gravitational axial anomaly contribution to the pion decay into two photons, is studied in detail in this context. In the process we show that the two-dimensional chiral pure gravitational Weyl anomaly does not receive noncommutative corrections. Pure gravitational chiral anomaly in $4k+2$ dimensions with matter fields being chiral fermions of spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$, is discussed and a noncommutative correction is found in both cases. Mixed anomalies are finally considered for both cases.

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1. Introduction

Noncommutative field theory has very intriguing new effects in quantum field theory as the UV/IR mixing discovered recently in Ref. [1], which in fact, has a stringy origin. Other nice surprise is the deep relation with string theory and M-theory [2,3] (for a nice review see, for instance, [4,5]). Another important effect in quantum field theory are the anomalies. Gauge anomalies, in particular axial and gauge chiral anomalies in noncommutative gauge theories has been discussed in a series of papers by various authors [6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25]. In particular, noncommutative gauge anomalies in noncommutative Yang-Mills theory was considered in Refs. [9,11,12,14,15] for planar diagrams with gauge group $U(N)$. For the case of nonplanar diagrams there has been some previous work in [10,26,27]. The analysis can be extended to other gauge groups by introducing the Seiberg-Witten map as in references [20,22,23,24,25].

On the other hand, recently various noncommutative theories of gravity have been proposed. In particular, the proposals in Refs. [28,29,30,31,32,33], different Moyal deformations of Einstein gravity in four dimensions are given. All these actions however are not manifestly invariant under the full noncommutative transformations since they are deformed under the Moyal product, with a constant noncommutativity parameter. Therefore they are not diffeomorphism invariant as far the Moyal product depend on the coordinate system. These products can be made diffeomorphism invariant, substituting the Moyal $*_M$-product by the Kontsevich $*_K$-product [34]. In this paper we will assume the uses of the Kontsevich $*_K$-product though we will avoid to use the subscript K.

Recently, another noncommutative proposals are given in Ref. [35,36]. In the former reference a manifestly SO(1,3) invariant noncommutative topological action for the gravitational theta terms is constructed. For appropriate boundary conditions give us the possibility to provide some insight about noncommutative gravitational instantons and noncommutative Lorentz gravitational anomalies. In the latter paper it was discussed a dynamical case of Einstein gravity by Moyal (or Kontsevich) deforming the self-dual projection of Einstein theory to find a manifestly SL(2,$\mathbb{C}$) invariant noncommutative theory.

Noncommutative topological actions which are the linear combination of the Euler number $\widehat{\chi}(X)$ and signature $\widehat{\sigma}(X)$, being an $\mathrm{SO}\widehat{(3,1)}$-invariant action, are very important since they describe the breakdown of chiral symmetry in the presence of gravitational fields. Due the technical difficulties in trying to make noncommutative the action $\chi(X)$

we propose a way by getting the noncommutative $\widehat{\mathrm{SL}(2,\mathbb{C})}$-invariant action $\widehat{\chi}(X)$, from a noncommutative version for the signature $\widehat{\sigma}(X)$, which is also, a $\widehat{\mathrm{SL}(2,\mathbb{C})}$-invariant action.

On the other hand, local gauge anomalies are associated to the lack of invariance of the fermionic one-loop effective action $\Gamma(Q) = \log\big[\det \not{D}\big]$, with $e^{-\Gamma} = \int \mathcal{D}\psi\mathcal{D}\psi^* e^{-\int_X L}$, under infinitesimal gauge transformations with the chiral matter fields $\psi(x)$ and $\psi^*(x)$ living in a complex representation Q of the gauge group G. In the case of theories gravitational couplings to matter there are different types of gravitational anomalies, depending on the type of transformations. Thus the Lorentz (or automorphisms) anomaly is related to the lack of gauge invariance of Γ under Lorentz transformations. When the symmetry group are the group of diffeomorphisms $Diff(X)$ of a smooth spacetime manifold X and $\Gamma(Q)$ is not generally covariant, then we have a diffeomorphism anomaly.

Gravitational corrections to the ABJ-anomaly was originally computed by Delbourgo and Salam [37] and further worked out in Refs. [38,39]. Soon after, gravitational anomalies were computed in a systematic way by Alvarez-Gaumé and Witten [40] (see also [41]). Through out the present paper we will not consider global gravitational anomalies [42].

In Ref. [35] it was argued about a noncommutative version of the the Lorentz group $\widehat{SO(4)}$ following a global procedure for computing chiral anomalies in gauge theory suggested by Harvey [43], which is based in the mathematical literature [44]. The application of these ideas to the diffeomorphism transformations connected to the identity, might predict new nontrivial noncommutative gravitational effects, which should be computed explicitly as a noncommutative correction to the gravitational contribution to the chiral anomaly.

In the present paper we compute gravitational axial and chiral anomalies starting from a of the full noncommutative gravitational theory and focus in the interaction lagrangian between chiral fermions and gravitons in the noncommutative spacetime. We will follow the t' Hooft's observation that the anomalies can be understood in terms of the low energy effective field theory and we will consider a noncommutative effective field theory describing the one-loop effective action of chiral fermions in background fields being the noncommutative gravitational field (pure gravitational anomalies) or/and noncommutative Yang-Mills fields (mixed anomalies). we will restrict to the computation of perturbative one-loop diagrams of chiral fermions with external gravitons or/and gluons in various dimensions. Only planar diagrams are considered in this paper.

This paper is organized as follows: In Section 2 we start with some arguments about some global aspects on noncommutative gravitational anomalies. In Section 3 we provide some basic features of perturbative noncommutative gravity and the corresponding noncommutative Feynman rules of the coupling of Weyl fermions to gravity. Section 4 is devoted to compute the noncommutative analog of Delbourgo-Salam axial gravitational anomaly which is the gravitational correction to the ABJ (axial) anomaly in four dimensions.

In Section 5 we discuss the gravitational gauge chiral anomaly in two dimensions. We show that in this case the noncommutative case coincides with the commutative one and there is not noncommutative correction. Section 6 is devoted to extend the computation of the one-loop amplitude to dimension $D = 4k + 2$. In here, after some preliminaries we compute the gravitational chiral gauge anomaly by evaluating directly the perturbative amplitude of the one loop diagram and the Schwinger procedure. In the same section is also computed the amplitude for spin-$\frac{3}{2}$ chiral fermions. In Section 7 we describe separately the mixed anomalies between gauge fields and gravitational fields coupled with spin-$\frac{1}{2}$ or spin-$\frac{3}{2}$ in a noncommutative space. Finally in Section 8 we give our final remarks.

2. Towards Noncommutative Gravitational Anomalies: Global Aspects

Before to proceed to compute gravitational anomalies in the noncommutative context we would like to make some global considerations about the nature of these anomalies.

From the topological perspective local gravitational anomalies are obtained through the computation of some suitable homotopy groups of the relevant gauge group.

In Ref. [35] it was argued about a noncommutative version of the the Lorentz group $\widehat{SO(4)}$ following a global procedure for computing chiral anomalies in gauge theory suggested by Harvey [43], which is based in the mathematical literature [44]. The proposal consists in assuming that $\widehat{SO(4)}$ consist of the set of compact orthogonal operator algebra $\mathbf{O}_{cpt}(\mathcal{H})$, defined on the separable real Hilbert space $\mathcal{H}$. The compactness property avoids the Kuiper theorem, which states that the set of pure orthogonal operators $\mathbf{O}(\mathcal{H})$ has trivial homotopy groups [44]. This algebra has non-trivial subalgebras which have the same homotopy than $SO(\infty)$ (up to Bott's periodicity 8), which may give rise to nontrivial new topological effects in noncommutative gravity. Noncommutative local Lorentz anomaly is

detected with $\pi_3(\mathbf{O}_{cpt}(\mathcal{H})) = \mathbb{Z}$. The choice of $\mathbf{O}_{cpt}(\mathcal{H})$ as a version of $\widehat{SO(4)} = SO(\infty)$ is highly not unique, thus there are many possibilities to do that and there is not a natural procedure to define $\widehat{SO(4)}$ and a more explicit way of computing the local gravitational anomaly is indeed needed.

In gravitational theories, the Lorentz group is only a part of the entire symmetry group. Thus, the moduli space of the pure gravity theory involves a richer phase space structure which consist of the quotient space: $\mathcal{M} = \mathcal{T}/\Gamma^+_\infty$, where $\mathcal{T} = Met(X)/Diff^+_0(X)$ is the Teichmüller space and Γ^+_∞ is the mapping class group given by the quotient group $\Gamma^+_\infty = Diff^+(X)/Diff^+_0(X)$. Here $Met(X)$ is the moduli space of Riemannian metrics on X, $Diff^+(X)$ is the group of all orientation preserving diffeomorphisms on X and $Diff^+_0(X)$ is the group of orientation preserving diffeomorphisms on X which are connected to the identity. However there is a restriction in the spacetime dimensionality in which the diffeomorphism anomaly can exists. This can exist only for $dim X = 4k + 2$ dimensions since only in $D = 4k + 2$ dimensions, the orthogonal group $O(1, D - 1)$ has complex representations.

Local gravitational anomalies in the usual commutative case appear when the mapping class group is the trivial group i.e., $\Gamma^+_\infty = 1$. Thus the moduli space is given by $\mathcal{M} = Met(X)/Diff^+_0(X)$. The global gravitational anomalies are related to the disconnectedness of Γ^+_∞, i.e. $\pi_0(\Gamma^+_\infty) \neq 1$. Now the moduli space for noncommutative gravity might be defined by $\widehat{\mathcal{M}} = \widehat{\mathcal{T}}/\widehat{\Gamma^+_\infty}$ with $\widehat{\mathcal{T}} = \widehat{Met(X)}/\widehat{Diff^+_0}(X)$ and $\widehat{\Gamma^+_\infty} = \widehat{Diff^+}(X)/\widehat{Diff^+_0}(X)$. Of course, on order to perform some computations on anomalies, one has to be able to provide suitable definitions for $\widehat{\Gamma^+_\infty}$, $\widehat{Diff^+}(X)$ and $\widehat{Diff^+_0}(X)$. Noncommutative local gravitational anomalies would arise when $\pi_2(\widehat{\mathcal{M}}) = \pi_1(\widehat{Diff^+_0}(X)) \neq 1$, where $\widehat{\mathcal{M}} = \widehat{Met(X)}/\widehat{Diff^+_0}(X)$.

Once again the choice of some suitable version of $\widehat{Diff^+_0}(X)$ as is highly not unique, there are many possibilities for it and there is not a natural procedure to define $\widehat{Diff^+_0}(X)$ and a more explicit way of computing the local gravitational anomaly is needed. In this paper we avoid the use the topological perspective and we will compute gravitational chiral anomalies by the direct and explicit computation of one-loop diagrams of chiral fermions coupled to external gravitons and/or gauge fields. In order to do that we will use the Feynman rules of a suitable noncommutative gravity given in the next section.

3. The Noncommutative Coupling of Gravity and Chiral Fermions

In this section we will give a brief overview of the pure noncommutative perturbative gravitational field and interaction with a noncommutative Weyl fermion. Our aim is to recall the relevant structure of couplings and Feynman rules, which we will need in the following sections.

As we reviewed in the introduction, at present there is not well established a definitive and well defined realistic noncommutative gravity theory. In this paper we will not deal with any specific noncommutative theory of gravity. This is because at the end we will not consider an specific theory of pure gravity but we will be interested only in the interactions of linearized noncommutative gravitational field to chiral fermions. However to be concrete we will briefly review a particular proposal of noncommutative Einstein gravity [29] given by the noncommutative Einstein-Hilbert action: $\widehat{I}_{EH} = -\frac{1}{16\pi G_N}\int_X d^4x(-e) * e^a_\mu(x) * e^b_\nu(x) * R^{\mu\nu}_{ab}(x)$, where $g_{\mu\nu}(x) = e^a_\mu(x) * e^b_\nu(x)\eta_{ab}$, and $R^{ab}_{\mu\nu}(x) = \partial_\mu\omega^{ab}_\nu(x) - \partial_\nu\omega^{ab}_\mu(x) + [\omega_\mu(x), \omega_\nu(x)]^{ab}_*$, with $\omega^{ab}_\mu(x)$, being the noncommutative spin connection and $[A,B]_* \equiv A * B - B * A$ is the Moyal bracket. Here $*$-product is defined by $F * G(x) \equiv \exp\left(\frac{i}{2}\Theta^{\mu\nu}\frac{\partial}{\partial y^\mu}\frac{\partial}{\partial z^\nu}\right)F(y)G(z)\Big|_{y=z=x}$. From now on in order to avoid causality problems we will take $\theta^{0\nu} = 0$.

Noncommutative perturbative gravity is defined as by a perturbative expansion $I = I^{(0)} + I^{(1)} + I^{(2)} + \mathcal{O}(\kappa^4)$ of the noncommutative Einstein-Hilbert action [32] generated by a perturbative expansion of the metric as follows $g_{\mu\nu} = \eta_{\mu\nu} - \kappa h_{\mu\nu} + \kappa^2 h^\alpha_\mu * h_{\alpha\nu} - \kappa^3 h^\alpha_\mu * h_{\alpha\beta} * h^\beta_\nu + \mathcal{O}(\kappa^4)$.

In Ref. [32] it was given the Feynman rules of this pure noncommutative gravity theory. In what follows we will give the corresponding Feynman rules governing the coupling of the noncommutative linear metric $h_{\mu\nu}(x)$ to chiral fermions.

3.1. Coupling Gravity to Chiral Fermions

Let's consider the theory in $D = 4k + 2$ dimensions. The coupling of gravitational field with chiral fermions is given as usual by

$$I_{int} = \int d^{4k+2}x \det(e) * e^{\mu a}(x) * \frac{1}{2}\overline{\psi}(x) * i\Gamma_a D_\mu\left(\frac{1-\overline{\Gamma}}{2}\right)\psi(x), \tag{3.1}$$

where D_μ is the covariant derivative with respect to the spin connection ω_μ^{ab} given by $D_\mu\psi(x) = \partial_\mu\psi(x) + \frac{1}{2}\omega_{\mu cd}\sigma^{cd}\psi(x)$, with $\sigma^{cd} = \frac{1}{4}[\Gamma^c, \Gamma^d]$, $\overline{\Gamma} = \Gamma_1 \dots \Gamma_{4k+2}$ and the Γ's are the Dirac matrices in euclidean $4k+2$ dimensions.

Our noncommutative action splits into two parts $I_{int} = I_1 + I_2$ where

$$I_1 = \frac{1}{2}\int dx\, \det(e) * e^{\mu a}(x) * \overline{\psi}(x) * i\Gamma_a \overset{\leftrightarrow}{\partial}_\mu \left(\frac{1-\overline{\Gamma}}{2}\right)\psi(x) \tag{3.2}$$

and

$$I_2 = \frac{1}{4}\int dx\, \det(e) * e^{\mu a}(x) * \omega_\mu^{cd}(x) * i\overline{\psi}(x) * \Gamma_{acd}\left(\frac{1-\overline{\Gamma}}{2}\right)\psi(x), \tag{3.3}$$

where $\Gamma_{acd} = \frac{1}{6}(\Gamma_a\Gamma_c\Gamma_d \pm \text{permutations})$.

The linearization of our noncommutative action I_{int} given by Eq. (3.1) leads to the Moyal deformation of linear gravity given by the lagrangian

$$L_1 = -\frac{1}{4}ih^{\mu\nu}(x) * \overline{\psi}(x) * \Gamma^\mu \overset{\leftrightarrow}{\partial}_\mu \left(\frac{1-\overline{\Gamma}}{2}\right)\psi(x), \tag{3.4}$$

and

$$L_2 = -\frac{1}{16}ih_{\lambda\alpha}(x) * \partial_\mu\overline{\psi}(x) * \Gamma^{\mu\lambda\nu}\left(\frac{1-\overline{\Gamma}}{2}\right)\psi(x). \tag{3.5}$$

The corresponding noncommutative Feynman rules can be reading from the lagrangians (3.4) and (3.5) and they are given by

$$-\frac{i}{4}\varepsilon^{\mu\nu}\Gamma_\mu\left(\frac{1-\overline{\Gamma}}{2}\right)(p+p')_\nu \exp\left(-\frac{i}{2}\Theta^{\mu\nu}p_\mu p'_\nu\right) \tag{3.6}$$

and

$$-\frac{i}{16}\Gamma^{\lambda\mu\nu}\left(\frac{1-\overline{\Gamma}}{2}\right)\varepsilon^{(1)}_{\nu\alpha}\varepsilon^{(2)}_{\lambda\alpha}\exp\big(\frac{i}{2}\Theta^{\rho\sigma}p_\rho p'_\sigma\big)$$

$$\times\Big[k_{1\mu}\exp\big(\frac{i}{2}\Theta^{\rho\sigma}k_{1\rho}k_{2\sigma}\big) - k_{2\mu}\exp\big(\frac{i}{2}\Theta^{\rho\sigma}k_{1\rho}k_{2\sigma}\big)\Big], \tag{3.7}$$

where $\varepsilon^{(i)}_{\mu\alpha}$ are the polarization tensors of the graviton field.

4. Noncommutative Delbourgo-Salam Gravitational Anomaly

Gravitational anomalies in four dimensions were studied first by Delbourgo and Salam [37] as a gravitational correction to the violation of a global symmetry responsible of the decay: $\pi^0 \to \gamma\gamma$. This idea was further developed in Refs. [38,39]. Here we shall discuss the noncommutative counterpart. Delbourgo and Salam [37] showed that in addition to the fermion triangle diagram with three currents, the triangle with one current J of a global symmetry and two energy-momentum tensors T is also anomalous. The corresponding contribution from the anomalous Ward identity is given by

$$\frac{1}{384\pi^2} R_{\kappa\lambda\rho\sigma} R^{\rho\sigma}_{\mu\nu} \varepsilon^{\kappa\lambda\mu\nu}. \tag{4.1}$$

This is precisely proportional to the signature invariant $\sigma(X)$ (or the first Pontrjagin class) which together the Euler number $\chi(X)$ are the classical topological invariants of the smooth spacetime manifold X.

Now we will discuss in detail the derivation of the noncommutative counterpart of Eq. (4.1). The scattering amplitude of the process in 4 dimensions is given by

$$\begin{aligned} &\mathrm{Tr} \int d^4p \{\Gamma \cdot p, \Gamma_{\kappa\lambda\mu\nu}\} \frac{\exp\big(-\frac{i}{2}\Theta^{\rho\sigma}(p-k_2)_\rho (p+k_1)_\sigma\big)}{[\Gamma\cdot(p+k_1)-M]} \varepsilon_{\rho_1\sigma_1} p^{\rho_1}\Gamma^{\sigma_1} \\ &\times \frac{\exp\big(-\frac{i}{2}\Theta^{\rho\sigma}(p+k_1)_\rho p_\sigma\big)}{(\Gamma\cdot p - M)} \varepsilon_{\rho_2\sigma_2} p^{\rho_2}\Gamma^{\sigma_2} \frac{\exp\big(-\frac{i}{2}\Theta^{\rho\sigma}p_\rho (p-k_2)_\sigma\big)}{[\Gamma\cdot(p-k_2)-M]}, \end{aligned} \tag{4.2}$$

where we have used the Feynman rule (3.6) in each vertex of the triangle diagram and the corresponding fermion propagators.

In order to evaluate this amplitude we promote the integral from 4 to 2ℓ dimensions

$$\begin{aligned} &\int d^{2\ell}p \frac{(\Gamma\cdot(p+k_1)+M)}{\big[(p+k_1)^2-M^2\big]} \cdot \exp\big(-\frac{i}{2}\Theta^{\rho\sigma}(p-k_2)_\rho (p+k_1)_\sigma\big) \varepsilon_{\rho_1\sigma_1} p^{\rho_1}\Gamma^{\sigma_1} \frac{(\Gamma\cdot p+M)}{\big[p^2-M^2\big]} \\ &\times \exp\big(-\frac{i}{2}\Theta^{\rho\sigma}(p+k_1)_\rho p_\sigma\big) \frac{(\Gamma\cdot(p-k_2)+M)}{\big[(p-k_2)^2-M^2\big]} \varepsilon_{\rho_2\sigma_2} p^{\rho_2}\Gamma^{\sigma_2} \cdot \exp\big(-\frac{i}{2}\Theta^{\rho\sigma}p_\rho (p-k_2)_\sigma\big) \end{aligned} \tag{4.3}$$

and as usual we introduce the Feynman's parameters

$$\frac{1}{ABC} \equiv 2\int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{\delta(1-x-y-z)}{(xA+yB+zC)^3}, \tag{4.4}$$

with $A=(p+k_1)^2-M^2$, $B=(p-k_2)^2-M^2$ and $C=p^2-M^2$. After a the redefinition of the momenta $p \to p' = p+k_1x-k_2y$ we find that $xA+yB+(1-x-y)C = p'^2+k_3^2xy-M^2$

and omitting the trace operation of Dirac matrices we find that $xA+yB+(1-x-y)C = p'^2 + k_3^2xy - M^2$ and

$$\int_0^1 dxdydz\ \delta(1-x-y-z)\int\frac{d^{2\ell}p}{(p^2+k_3^2xy-M^2)^3}$$
$$\times\mathrm{Tr}\bigg\{\{\Gamma\cdot p,\Gamma_{\kappa\lambda\mu\nu}\}\big[\Gamma\cdot(p+zk_1-xk_3)+M\big](p+xk_2)^{\rho_1}\Gamma^{\sigma_1}$$
$$\times\big[\Gamma\cdot(p+xk_2-yk_1)+M\big](p-yk_1)^{\rho_2}\Gamma^{\sigma_2}\big[\Gamma\ (p+yk_3-zk_2)+M\big]\bigg\}\exp\bigg(-\frac{i}{2}\Theta^{\rho\sigma}k_{1\rho}k_{2\sigma}\bigg). \tag{4.5}$$

Here we have redefined once again $p'\to p$ and we have performed the sum of the contributions of the phases in the noncommutative parameter Θ. Integrating out the variable z and keeping only the divergent terms we finally get

$$2k_2^{\rho_1}k_1^{\rho_2}\int_0^1 dxdy\ \theta(1-x-y)xy\int\frac{d^{2\ell}p}{(p^2+k_3^2xy-m^2)^3}$$
$$\times\mathrm{Tr}(\{\Gamma\cdot p,\Gamma_{\kappa\lambda\mu\nu}\}\Gamma\cdot p\ \Gamma^{\sigma_1}\Gamma^{\sigma_2}\Gamma\cdot k_1\Gamma\cdot k_2)\exp\bigg(-\frac{i}{2}\Theta^{\rho\sigma}k_{1\rho}k_{2\sigma}\bigg), \tag{4.6}$$

where $\theta(x)$ is the Heavside function. Now we proceed to compute the trace of products of gamma matrices using the cyclic property of the trace by using the identity $\mathrm{Tr}\big(\Gamma_{\kappa\lambda\mu\nu}\Gamma^{\sigma_1}\Gamma^{\sigma_2}\Gamma^{\alpha}\Gamma^{\beta}\big) = 2^\ell\delta^{[\sigma_1}_{[\kappa}\delta^{\sigma_2}_{\lambda}\delta^{\alpha}_{\mu}\delta^{\beta]}_{\nu]} = 2^\ell\varepsilon^{\sigma_1\sigma_2\alpha\beta}\varepsilon_{\kappa\lambda\mu\nu}$ we finally obtain

$$2k_2^{\rho_1}k_1^{\rho_2}\int_0^1 dxdy\ \theta(1-x-y)xy\int\frac{d^{2\ell}p}{(p^2+k_3^2xy-m^2)^3}2\bigg(\frac{\ell-2}{\ell}\bigg)$$
$$\times\mathrm{Tr}\bigg[\{\Gamma\cdot p,\Gamma_{\kappa\lambda\mu\nu}\}\Gamma^{\sigma_1}\Gamma^{\sigma_2}\Gamma\cdot k_1\Gamma\cdot k_2+\dots\bigg]\exp\bigg(-\frac{i}{2}\Theta^{\rho\sigma}k_{1\rho}k_{2\sigma}\bigg) \tag{4.7}$$

or in other form

$$2^{\ell+1}k_2^{\rho_1}k_1^{\rho_2}\int_0^1 dxdy\ \theta(1-x-y)xy\int\frac{d^{2l}p}{(p^2+k_3^2xy-m^2)^3}2\bigg(\frac{\ell-2}{\ell}\bigg)$$
$$\times k_{1\alpha}k_{2\beta}\varepsilon^{\sigma_1\sigma_2\alpha\beta}\varepsilon_{\kappa\lambda\mu\nu}\exp\bigg(-\frac{i}{2}\Theta^{\rho\sigma}k_{1\rho}k_{2\sigma}\bigg). \tag{4.8}$$

In the integration on the momentum p we have used the following identity

$$\int d^{2\ell}p\frac{p^2}{(p^2+k_3^2xy-M^2)^3} = \frac{i\pi^\ell}{(k_3^2xy-M^2)^{3-\ell}}\frac{\Gamma(2-\ell)}{\Gamma(3)}\ell(k_3^2xy-M^2). \tag{4.9}$$

Then finally we obtain

$$2^{\ell+1}(\ell-2)k_2^{\rho_1}k_1^{\rho_2}\varepsilon^{\sigma_1\sigma_2\alpha\beta}\varepsilon_{\kappa\lambda\alpha\beta}k_{1\alpha}k_{2\beta}\exp\bigg(-\frac{i}{2}\Theta^{\rho\sigma}k_{1\rho}k_{2\sigma}\bigg)$$

$$\times(4\pi)^{-\ell}\Gamma(2-\ell)\int(k_3^2xy-M^2)^{\ell-2}ixy\;\theta(1-x-y)dxdy+\ldots. \tag{4.10}$$

Performing the expansion of the gamma function $\Gamma(\varepsilon)$ for small values of ε with $\varepsilon=2-\ell$, taking the limit $\ell\to 2$ and evaluating the integral in x and y we finally get

$$-i\frac{k_2^{\rho_1}k_1^{\rho_2}}{12\pi^2}\varepsilon^{\sigma_1\sigma_2\alpha\beta}\varepsilon_{\kappa\lambda\alpha\beta}k_{1\alpha}k_{2\beta}\exp\big(-\frac{i}{2}\Theta^{\rho\sigma}k_{1\rho}k_{2\sigma}\big). \tag{4.11}$$

Taking into account the most general Lorentz invariant amplitude we get

$$-\frac{i}{192\pi^2}\varepsilon_{\rho_1\sigma_1}(k_1)\varepsilon_{\rho_2\sigma_2}(k_2)k_{1\alpha}k_{2\beta}\varepsilon^{\sigma_1\sigma_2\alpha\beta}\big(\eta^{\rho_1\rho_2}k_1\cdot k_2-k_1^{\rho_2}k_2^{\rho_1}\big)\varepsilon_{\kappa\lambda\alpha\beta}\exp\bigg(-\frac{i}{2}\Theta^{\rho\sigma}k_{2\rho}k_{1\sigma}\bigg). \tag{4.12}$$

In the coordinate space this expression can be rewritten as

$$\varepsilon^{\sigma_1\sigma_2\alpha\beta}\bigg(\partial_\alpha\partial_\gamma h_{\rho_1\sigma_1}*\partial_\beta\partial^\gamma h^{\rho_1}_{\sigma_2}-\partial_\alpha\partial^{\rho_2}h_{\rho_1\sigma_1}*\partial_\beta\partial^{\rho_1}h_{\rho_2\sigma_2}\bigg)\varepsilon_{\kappa\lambda\alpha\beta}. \tag{4.13}$$

This equation can be rewritten as

$$\frac{1}{384\pi^2}R_{\kappa\lambda\rho\sigma}*R^{\rho\sigma}_{\mu\nu}\varepsilon^{\kappa\lambda\mu\nu}. \tag{4.14}$$

This is precisely the noncommutative signature invariant $\widehat{\tau}(X)=\int R*\widetilde{R}\;d^4x$, where the tilde over R stands for the Hodge dual with respect the tangent space indices. Compare this with the noncommutative signature $\widehat{\sigma}(X)$ of Ref. [35] where the Hodge duality was associated to the tetrad indices.

5. Noncommutative Pure Gravitational Anomaly in Two Dimensions

In the previous section we have introduced Feynman rules for noncommutative perturbative quantum gravity relevant to compute the chiral gravitational anomalies. Before to compute the noncommutative gravitational anomaly in $D=4k+2$ dimensions in this section we are going into the details of the computation of the pure gravitational anomaly in two dimensions. We will follow the notation and conventions of Ref. [40].

In two dimensions the noncommutative action for a Majorana-Weyl fermion in a gravitational field is given by $I = \int d^2x \, det(e) * e^{\mu a}(x) * \frac{1}{2}\overline{\psi}(x) * i\Gamma_a \partial_\mu \psi(x)$. At the linearized level, the lagrangian is given by

$$L_{int} = -\frac{1}{4} h^{\mu\nu}(x) * i\overline{\psi}(x) * \Gamma_\mu \partial_\nu \psi(x). \tag{5.1}$$

The corresponding energy-momentum tensor is given by

$$T_{\mu\nu}(x) = \frac{1}{2} i\overline{\psi}(x) * \Gamma_\mu \partial_\nu \psi(x). \tag{5.2}$$

In the light-cone coordinates $x^{\pm} = \frac{1}{\sqrt{2}}(x^0 \pm x^1)$, Dirac matrices are decomposed into $\Gamma^{\pm} = \frac{1}{\sqrt{2}}(\Gamma^0 + \Gamma^1)$, with $(\Gamma^{\pm})^2 = 0$ and $\Gamma^+\Gamma^- + \Gamma^-\Gamma^+ = 2$. In these coordinates the energy-momentum tensor is given by

$$T_{++}(x) = \frac{1}{2} i\overline{\psi}(x) * \Gamma_+ \partial_+ \psi(x), \tag{5.3}$$

while the interaction action (5.1) of the gravitational field with fermions in the light-cone coordinates reduces to

$$L_{int} = -\frac{1}{4} i h_{--}(x) * \overline{\psi}(x) * \Gamma_+ \partial_+ \psi(x), \tag{5.4}$$

then only the component $h_{--}(x)$ of the graviton is coupled to chiral matter described by the the component $T_{++}(x)$ of the energy-momentum tensor. The effective action to second order in the metric perturbation h is encoded in the two-point correlation function

$$U(p) = \int d^2x \exp\big(ip \cdot x\big) \langle \Omega | T(T_{++}(x) * T_{++}(0) | \Omega \rangle, \tag{5.5}$$

where

$$\langle \Omega | T(T_{++}(x) * T_{++}(0) | \Omega \rangle = -\frac{1}{4} \int \prod_{i=1}^{2} \frac{d^2 q_i}{(2\pi)^2} \prod_{j=1}^{2} \frac{d^2 q'_i}{(2\pi)^2} \langle \widetilde{\overline{\psi}}(q_1) \gamma_+ \partial_+ \psi(q'_1) \cdot \widetilde{\overline{\psi}}(q_2) \gamma_+ \partial_+ \psi(q'_2) \rangle$$

$$\times \exp\big(i(q_1 - q'_1)x\big) \exp\big(i(q_2 - q'_2)x\big) \exp\left(i \frac{\Theta_{\rho\sigma}}{2} \sum_{j=1}^{3} q_j^\rho q_j'^\sigma \right). \tag{5.6}$$

The naive Ward identity is given by $p_- U(p) = 0$. This should imply $U(p) = 0$ for all p_-, thus it should be an anomaly. Thus we can compute $U(p)$ by evaluating the corresponding one-loop diagram with two external gravitons, this yields

$$U(p) = -\frac{1}{4}\int \frac{dk_+ dk_-}{(2\pi)^2}(2k+p)_+^2 \frac{k_+ \exp\big(-\frac{i}{2}\Theta^{\rho\sigma}k_\rho p_\sigma\big)}{k_+k_- + i\varepsilon}$$

$$\times \frac{(k+p)_+ \exp\big(-\frac{i}{2}\Theta^{\rho\sigma}p'_\rho k_\sigma\big)}{(k+p)_+(k+p)_- + i\varepsilon}\delta(p+p')\cdot \exp\big(i(p+p')x\big)$$

$$= -\frac{1}{4}\int \frac{dk_+ dk_-}{(2\pi)^2}(2k+p)_+^2 \frac{1}{k_- + i\varepsilon/k_+}\frac{\exp\big(-\frac{i}{2}\Theta^{\rho\sigma}p'_\rho p_\sigma\big)}{(k+p)_- + i\varepsilon/(k+p)_+}\cdot\delta(p+p')\cdot\exp\big(i(p+p')x\big), \tag{5.7}$$

where we have used the Feynman rule (3.6) to compute $U(p)$.

In light-cone coordinates the Moyal product is given by $\exp\big(-\frac{i}{2}\Theta^{\rho\sigma}p'_\rho p_\sigma\big) = \exp\big(-\frac{1}{2}\Theta^{+-}(p'_+p_- - p'_-p_+)\big)$. Thus by analytic methods the computation of the integrals gives

$$U(p) = \frac{i}{8\pi}\int_{-p_+}^{0} dk_+ \frac{(2k+p)_+^2}{p_-}\exp\big(-\frac{i}{2}\Theta^{\rho\sigma}p'_\rho p_\sigma\big)\delta(p+p')$$

$$= \frac{i}{24\pi}\frac{p_+^3}{p_-}\exp\bigg(-\frac{i}{2}\Theta^{\rho\sigma}p'_\rho p_\sigma\bigg)\exp\big(i(p+p')x\big)\delta(p+p'). \tag{5.8}$$

Thus the anomalus gravitational Ward identity is given by

$$p_-U(p) = \frac{i}{24\pi}p_+^3\exp\bigg(-\frac{i}{2}\Theta^{\rho\sigma}p'_\rho p_\sigma\bigg)\exp\big(i(p+p')x\big)\delta(p+p'). \tag{5.9}$$

The computation of the two-graviton diagram coupled with chiral fermions in the noncommutative theory is given by the effective action

$$L_+^{eff}(h_{\mu\nu}) = -\frac{1}{192\pi}\int d^2p d^2p' \frac{p_+^3}{p_-}h_{--}(p)$$

$$\times \exp\bigg(-\frac{i}{2}\Theta^{\rho\sigma}p'_\rho p_\sigma\bigg)h_{--}(p')\exp\big(i(p+p')x\big)\delta(p+p'). \tag{5.10}$$

Similarly to the usual commutative case there is no way to add generic counterterms ΔL_+^{eff} such that $L_+^{eff} + \Delta L_+^{eff}$ be invariant under general coordinate transformations.

Thus, let us consider a Dirac fermion in $1+1$ dimensions, then we have the corresponding action L_D^{eff} is the superposition of L_+^{eff} and its corresponding parity conjugate L_-^{eff} resulting

$$L_D^{eff}(h_{\mu\nu}) = -\frac{1}{192\pi}\int d^2p d^2p' \bigg[\frac{p_+^3}{p_-}h_{--}(p)\exp\bigg(-\frac{i}{2}\Theta^{\rho\sigma}p'_\rho p_\sigma\bigg)h_{--}(p')$$

$$+\frac{p_-^3}{p_+}h_{++}(p)\exp\bigg(-\frac{i}{2}\Theta^{\rho\sigma}p'_\rho p_\sigma\bigg)h_{++}(p')\bigg]\exp\big(i(p+p')x\big)\delta(p+p'). \tag{5.11}$$

This action is not invariant under infinitesimal general coordinate transformations $\delta x^\mu = \varepsilon^\mu$, $h_{\mu\nu}$ transforms as $\delta h_{\mu\nu}(x) = -\partial_\mu \varepsilon_\nu(x) - \partial_\nu \varepsilon_\mu(x)$ or in the momentum space

$$\delta h_{++}(p) = -2ip_+\varepsilon_+, \qquad \delta h_{+-}(p) = -ip_-\varepsilon_+ - ip_+\varepsilon_-, \quad \delta h_{--}(p) = -2ip_-\varepsilon_-. \tag{5.12}$$

However in this case there exist a counterterm ΔL_D^{eff} which can be added to L_D^{eff} such that it become invariant under general coordinate transformations

$$\Delta L_D^{eff} = -\frac{1}{192\pi}\int d^2pd^2p'\bigg[\frac{p_+^3}{p_-}h_{--}(p)\exp\Big(-\frac{i}{2}\Theta^{\rho\sigma}p'_\rho p_\sigma\Big)h_{--}(p')$$

$$+\frac{p_-^3}{p_+}h_{++}(p)\exp\Big(-\frac{i}{2}\Theta^{\rho\sigma}p'_\rho p_\sigma\Big)h_{++}(p') + 2p_+p_-h_{++}(p)\exp\Big(-\frac{i}{2}\Theta^{\rho\sigma}p'_\rho p_\sigma\Big)h_{--}(p')$$

$$-4p_+^2h_{--}(p)\exp\Big(-\frac{i}{2}\Theta^{\rho\sigma}p'_\rho p_\sigma\Big)h_{+-}(p') - 4p_-^2h_{++}(p)\exp\Big(-\frac{i}{2}\Theta^{\rho\sigma}p'_\rho p_\sigma\Big)h_{+-}(p')$$

$$+4p_+p_-h_{+-}(p)\exp\Big(-\frac{i}{2}\Theta^{\rho\sigma}p'_\rho p_\sigma\Big)h_{+-}(p')\bigg]\delta(p+p'). \tag{5.13}$$

It is easy to see that this action can be rewritten in a compact form as following

$$\Delta L_D^{eff} = -\frac{1}{192\pi}\int d^2pd^2p'\frac{R(p)\exp\big(-\frac{i}{2}\Theta^{\rho\sigma}p'_\rho p_\sigma\big)R(p')}{p_+p_-}\delta(p+p'), \tag{5.14}$$

which after integration in the variable p' gives the usual correction to the commutative counterpart of (5.13).

$$\Delta L_D^{eff} = -\frac{1}{192\pi}\int d^2p\frac{R(p)R(-p)}{p_+p_-}. \tag{5.15}$$

where $R(p)$ is the linearized term of the noncommutative curvature scalar which is given by $R(p) = p_+^2h_{--} + p_-^2h_{++} - 2p_+p_-h_{+-}$.

There a quantum correction to $T_{+-}(p) = 0$ which holds classically due to the introduction of h_{+-} in the counterterm lagrangian ΔL_D^{eff} and we have an expectation value of T_{+-} different to zero which gives rise to the gravitational anomaly

$$\langle 2T_{+-}(p)\rangle = -2\frac{\delta\Delta L_D^{eff}}{\delta h_{+-}(-p)} = -\frac{1}{24\pi}R(p). \tag{5.16}$$

By momentum conservation we have in the above analysis that $p' = -p$ through the $\delta(p+p')$ and the phase factor $\exp\big(-i\Theta^{\rho\sigma}p'_\rho p_\sigma\big)$ is equal to one and therefore there is no a modification to the gravitational anomaly in two dimensions in a noncommutative space.

6. Noncommutative Gravitational Anomalies in $D = 4k+2$ dimensions

6.1. Preliminaries

In this subsection we compute the one-loop diagram of $\frac{D}{2}+1 = 2k+2$ external gravitons of momentum $p_\mu^{(i)}$ and polarizations $\varepsilon_{\mu\nu}^{(i)}$ with $i = 1, \ldots, 2k+2$. In this computation we will follow Ref. [40] by using Adler's prescription of an equivalent diagram with $2k+1$ external gravitons with only one insertion of an axial factor $\frac{1}{2}(1-\overline{\Gamma})$ and $2k+1$ non-anomalous vertices. In what follows we assume that the polarization tensor $\varepsilon_{\mu\nu}$ given by $\varepsilon_{\mu\nu} = i(p_\mu \varepsilon_\nu + p_\nu \varepsilon_\mu)$ (where ε_μ is the parameter involved in the coordinate transformation $x^\mu \to x^\mu + \varepsilon^\mu$) will be factorized as: $\varepsilon_{\mu\nu}^{(i)} = \varepsilon_\mu^{(i)} \cdot \varepsilon_\nu^{(i)}$.

Then the total one-loop amplitude is proportional to:

$$\begin{aligned}
\mathcal{A} \propto \mathrm{Tr}\Bigg[\overline{\Gamma} \exp\bigg(-\frac{i}{2}\Theta^{\rho\sigma} k_\rho (k - p^{(1)} - \ldots - p^{(2k+1)})_\sigma\bigg)(\not{k} + M) \\
\times \not{\varepsilon}^{(1)} \exp\bigg(-\frac{i}{2}\Theta^{\rho\sigma}(k-p^{(1)})_\rho k_\sigma\bigg)(\not{k} - \not{p}^{(1)} + M)\, \not{\varepsilon}^{(2)} \exp\bigg(-\frac{i}{2}\Theta^{\rho\sigma}(k-p^{(1)}-p^{(2)})_\rho (k-p^{(1)})_\sigma\bigg) \\
\times(\not{k} - \not{p}^{(1)} - \not{p}^{(2)} + M)\, \not{\varepsilon}^{(3)} \exp\bigg(-\frac{i}{2}\Theta^{\rho\sigma}(k - p^{(1)} - p^{(2)} - p^{(3)})_\rho (k - p^{(1)} - p^{(2)})_\sigma\bigg) \\
\ldots \times \not{\varepsilon}^{(2k+1)} \exp\bigg(-\frac{i}{2}\Theta^{\rho\sigma}(k - p^{(1)} - \ldots - p^{(2k+1)})_\rho (k - p^{(1)} - \ldots - p^{(2k)})_\sigma\bigg) \\
\times(\not{k} - \not{p}^{(1)} - \ldots - \not{p}^{(2k+1)} + M)\Bigg],
\end{aligned} \tag{6.1}$$

where we have used the Feynman rule (3.6) in each non-anomalous vertex. In the amplitude we have omitted a $(2k+1)$ factors of the form $\frac{1}{p^2 - M^2}$ in each non-anomalous vertex.

Now, in order eliminate Dirac matrices we require that $\mathrm{Tr}\big(\overline{\Gamma}\Gamma_{\mu_1}\Gamma_{\mu_2}\ldots\Gamma_{\mu_{4k+2}}\big) = -2^{2k+1}\varepsilon_{\mu_1\mu_2\ldots\mu_{4k+2}}$. Thus we can factorize the dependence on the noncommutativity parameter Θ

$$\mathcal{A} \propto 2^{2k+1} M R\big(\varepsilon^{(i)}, p^{(j)}\big), \tag{6.2}$$

where $R(\varepsilon^{(i)} p^{(j)})$ is a kinematical factor which depends only on the external momenta and polarization vectors

$$R(\varepsilon^{(i)}, p^{(j)}) = -\varepsilon_{\mu_1\mu_2\ldots\mu_{4k+2}} p^{(1)}_{\mu_1} \varepsilon^{(1)}_{\mu_2} p^{(2)}_{\mu_3} \varepsilon^{(2)}_{\mu_4} \ldots p^{(2k+1)}_{\mu_{4k+1}} \varepsilon^{(2k+1)}_{\mu_{4k+2}}, \tag{6.3}$$

leaving $R(\varepsilon^{(i)}, p^{(j)})$ independent on Θ.

Using the Feynman rule (3.6) in each one of the $2k+1$ vertices we have for the i-th vertex there is the insertion of a factor: $-\frac{1}{4}i\varepsilon^{(i)}_{\mu}(p+p')^{\mu}\frac{1}{p^2-M^2}\exp\big(-\frac{i}{2}\Theta^{\rho\sigma}p_{\rho}p'_{\sigma}\big)$, where p is the incoming momentum and p' is the outgoing momentum. The whole contribution is encoded in the amplitude $\mathcal{Z}(\varepsilon^{(i)},p^{(j)},\Theta)$. The total amplitude is then given by

$$I_{\frac{1}{2}} = 2^{2k+1}M^2 R(\varepsilon^{(i)},p^{(j)})\cdot \mathcal{Z}(\varepsilon^{(i)},p^{(j)},\Theta), \tag{6.4}$$

where $\mathcal{Z}$ can be reinterpreted as the amplitude for a charged scalar field of mass M and charge $\frac{1}{4}$ in a loop coupled to $(2k+2)$ photons of momenta $p^{(j)}$ and polarization tensors $\varepsilon^{(i)}$ in a noncommutative spacetime.

Then all the information of the noncommutative parameter is in the amplitude $\mathcal{Z}$ and we need to find a way of computing the amplitude

$$\mathcal{Z}(\varepsilon^{(i)},p^{(j)},\Theta) = \int d^{4k+2}k \frac{\prod_{j=1}^{2k+2}\exp\bigg(-\frac{i}{2}\Theta^{\rho\sigma}\sum_j l^{(j)}_{\rho}l^{(j+1)}_{\sigma}\bigg)\varepsilon\cdot(l_j+l_{j+1})}{\prod_{j=1}^{2k+2}(l_j^2-M^2)}. \tag{6.5}$$

As in the commutative case, this problem can be carried over to the residual problem of the computation of this amplitude for a one-loop diagram with $2k+2$ external photons interacting noncommutatively with a massive complex scalar field of charge $\frac{1}{4}$ with propagators $i/(p^2-M^2)$ under the condition that in the i-th vertex we have a factor $-\frac{1}{4}i\varepsilon^{(i)}_{\mu}(p+p')^{\mu}\exp\big(-\frac{i}{2}\Theta^{\rho\sigma}p_{\rho}p'_{\sigma}\big)$, where p and p' [40]. This problem was discussed by Schwinger [45] for the commutative case and used by [40] to compute $\mathcal{Z}$. In this paper we follow the same strategy for the noncommutative case. In the next subsection, we give the details of the explicit computation of this noncommutative residual interaction. Basically we will have a noncommutative one of this interaction in each non-anomalous vertex and we find an exact solution for it and then apply it to compute $\mathcal{Z}$.

6.2. Explicit Computation of the Noncommutative Residual Interaction

We start from a theory for a complex scalar field of mass M coupled to an abelian gauge field in a noncommutative space. Due to the noncommutative bosonization, this system will be equivalent to a Schwinger model. The Schwinger model has been discussed in the noncommutative context in [46,47,48,49], however in the present paper we follow a different procedure. Consider the following action

$$L=\int d^{2p}x\big(D^{\mu}\overline{\phi}*D_{\mu}\phi+M^2\overline{\phi}*\phi\big), \tag{6.6}$$

with $D_{\mu}\phi=\partial_{\mu}\phi-ieA_{\mu}*\phi$ and $D_{\mu}\overline{\phi}=\partial_{\mu}\overline{\phi}+ie\overline{\phi}*A_{\mu}$. If we use the definition for star product $(f*g)(x)=fe^{\overleftarrow{\partial}_{\alpha}\Theta^{\alpha\beta}\overrightarrow{\partial}_{\beta}}g(x)$, where $\overleftarrow{\partial}_{\alpha}\Theta^{\alpha\beta}\overrightarrow{\partial}_{\beta}\equiv\frac{i}{2}\Theta^{\alpha\beta}\overleftarrow{\partial}_{\alpha}\overrightarrow{\partial}_{\beta}$. Some results found by Schwinger [45] were used in Ref. [40] as an tool to compute the gravitational anomaly in $4k+2$ dimensions for gravitons coupled to spin-$\frac{1}{2}$ fields.

The first term of the RHS of (6.6) noncommutative action is given by

$$\int d^{2p}xD^{\mu}\overline{\phi}*D_{\mu}\phi=\int\Big(\partial^{\mu}\overline{\phi}+ie\overline{\phi}e^{\overleftarrow{\partial}_{\alpha}\Theta^{\alpha\beta}\overrightarrow{\partial}_{\beta}}A^{\mu}\Big)\Big(\partial_{\mu}\phi-ieA_{\mu}e^{\overleftarrow{\partial}_{\alpha}\Theta^{\alpha\beta}\overrightarrow{\partial}_{\beta}}\phi\Big), \tag{6.7}$$

where we have used the cyclicity property of the trace $\int dxf(x)*g(x)=\int dxf(x)g(x)$ for any f and g. Expanding RHS of this expression and integrating by parts, we can factorize it as:

$$\int\overline{\phi}\Big\{-\Big[\partial^{\mu}-ie\Big(e^{\overleftarrow{\partial}_{\alpha}\Theta^{\alpha\beta}\overrightarrow{\partial}_{\beta}}A^{\mu}\Big)\Big]\Big[\partial_{\mu}-ie\Big(A_{\mu}e^{\overleftarrow{\partial}_{\alpha}\Theta^{\alpha\beta}\overrightarrow{\partial}_{\beta}}\Big)\Big]\Big\}\phi. \tag{6.8}$$

Then the starting action (6.6) results:

$$L=\int\overline{\phi}\Big\{-\Big[\partial^{\mu}-ie\Big(e^{\overleftarrow{\partial}_{\alpha}\Theta^{\alpha\beta}\overrightarrow{\partial}_{\beta}}A^{\mu}\Big)\Big]\Big[\partial_{\mu}-ie\Big(A_{\mu}e^{\overleftarrow{\partial}_{\alpha}\Theta^{\alpha\beta}\overrightarrow{\partial}_{\beta}}\Big)\Big]+M^2\Big\}\phi. \tag{6.9}$$

Let us define the euclidean partition function for a complex massive scalar field propagating in a constant electromagnetic field as:

$$Z=\int[\mathcal{D}\phi][\mathcal{D}\overline{\phi}]\exp\big(-L\big), \tag{6.10}$$

then the effective action Γ is related to the partition function in the form $Z=e^{-\Gamma}$

$$\Gamma=\mathrm{Tr}\ln\Big\{-\Big[\partial^{\mu}-ie\Big(e^{\overleftarrow{\partial}_{\alpha}\Theta^{\alpha\beta}\overrightarrow{\partial}_{\beta}}A^{\mu}\Big)\Big]\Big[\partial_{\mu}-ie\Big(A_{\mu}e^{\overleftarrow{\partial}_{\alpha}\Theta^{\alpha\beta}\overrightarrow{\partial}_{\beta}}\Big)\Big]+M^2\Big\}. \tag{6.11}$$

The Schwinger representation for the logaritmic function can be expressed as follows:

$$\Gamma=-\mathrm{Tr}\int_0^{\infty}\frac{ds}{s}\Big\{e^{-s\Big[-\Big(\partial^{\mu}-ie\big(e^{\overleftarrow{\partial}_{\alpha}\Theta^{\alpha\beta}\overrightarrow{\partial}_{\beta}}A^{\mu}\big)\Big)\Big(\partial_{\mu}-ie\big(A_{\mu}e^{\overleftarrow{\partial}_{\alpha}\Theta^{\alpha\beta}\overrightarrow{\partial}_{\beta}}\big)\Big)+M^2\Big]}-e^{-s}\Big\}. \tag{6.12}$$

We can write the strength field $F_{\mu\nu}$ of the constant electromagmnetic field as usual in the Schwinger representation, as a diagonal block matrix and work only with a generic block of two components. Also we will take the following gauge:

$$A_1 = 0, \qquad A_2 = Fx^1. \tag{6.13}$$

Notice that in this gauge, higher order terms than the first order vanish in the expansion of the Moyal product. With this in mind we can calculate the effective action Γ as:

$$\Gamma = -\mathrm{Tr}\int_0^\infty \frac{ds}{s}e^{-sM^2}\exp\Bigg\{s\Bigg[(\partial^\mu - ieA^\mu)(\partial_\mu - ieA_\mu) - ie\frac{i}{2}\Theta^{\alpha\beta}(\partial^\mu\partial_\alpha A_\mu\partial_\beta + \partial_\alpha A_\mu\partial^\mu\partial_\beta)$$

$$-ie\frac{i}{2}\Theta^{\alpha\beta}(\partial_\alpha\partial_\beta A^\mu\partial_\mu + \partial_\beta A^\mu\partial_\alpha\partial_\mu) - e^2\Bigg(\frac{i}{2}\Theta^{\alpha\beta}(A^\mu\partial_\alpha A_\mu\partial_\beta - \partial_\alpha\partial_\beta A^\mu A_\mu - \partial_\beta A^\mu\partial_\alpha A_\mu)$$

$$-\left(\frac{i}{2}\right)^2\Theta^{\alpha\beta}\Theta^{\lambda\delta}(\partial_\alpha\partial_\beta A^\mu\partial_\lambda A_\mu\partial_\delta + \partial_\beta A^\mu\partial_\alpha\partial_\lambda A_\mu\partial_\delta + \partial_\beta A^\mu\partial_\lambda A_\mu\partial_\alpha\partial_\delta)\Bigg)\Bigg]\Bigg\} - e^{-s}. \tag{6.14}$$

Now we focus in the operator in the exponential given by

$$\Big[\partial^\mu - ie\big(e^{\overleftarrow{\partial}_\alpha\Theta^{\alpha\beta}\overrightarrow{\partial}_\beta}A^\mu\big)\Big]\Big[\partial_\mu - ie\Big(A_\mu e^{\overleftarrow{\partial}_\alpha\Theta^{\alpha\beta}\overrightarrow{\partial}_\beta}\Big)\Big],$$

it is necessary to consider only one two-dimensional subspace expanded by the corresponding block. After some simplifications we obtain that this operator gives $\partial_1^2 + \partial_2^2 - ieFx^1\partial_2 + i^2e^2F^2(x^1)^2 + e\Theta F\partial_2^2 - \frac{i}{2}e^2\Theta F^2x^1\partial_2 + \frac{1}{4}e^2\Theta^2F^2\partial_2^2$. Thus finally it factorizes as

$$\partial_1^2 + \Big[\big(1 + \frac{e\Theta F}{2}\big) - ieFx^1\Big]^2. \tag{6.15}$$

Now, after substitute $\widehat{p}_j = -i\partial_j$ in this last expression we get

$$-\Big\{\widehat{p}_1^2 + \Big[\big(1 + \frac{e\Theta F}{2}\big)\widehat{p}_2 - eF\widehat{x}^1\Big]^2\Big\}. \tag{6.16}$$

Then, in order to get the efective action (6.12) we need to compute alternatively

$$I = \mathrm{Tr}\exp\Big(-s\Big\{\widehat{p}_1^2 + \Big[\big(1 + \frac{e\Theta F}{2}\big)\widehat{p}_2 - eF\widehat{x}^1\Big]^2\Big\}\Big) \tag{6.17}$$

Considering the problem in a box of volume $L \times L$ and using the definition of the trace we finally get

$$I = \left(\int dx_2 \int \frac{dp_2}{2\pi}\right) \mathrm{Tr}_1 \exp\left(-s\left\{\widehat{p}_1^2 + e^2F^2\left[\widehat{x}^1 - \left(\frac{1}{eF} + \frac{\Theta}{2}\right)p_2\right]^2\right\}\right). \tag{6.18}$$

Thus, we obtain the effective action of a one-dimensional harmonic oscillator Then $\left(\frac{1}{eF} + \frac{\Theta}{2}\right)p_2$ with effective center at $(x_1)_0 = \left(\frac{1}{eF} + \frac{\Theta}{2}\right)p_2$. Boundary condition: $0 \leq p_2 \leq \left(\frac{1}{eF} + \frac{\Theta}{2}\right)^{-1} L$ implies that

$$I = (VolR^2)\frac{1}{2\pi}\left(\frac{eF}{1 + \frac{\Theta eF}{2}}\right)\mathrm{tr}_y \exp\{-s(\widehat{p}_y^2 + e^2F^2\widehat{y}^2)\} \tag{6.19}$$

where $L = VolR$. This trace yields precisely the partition function of an ordinary harmonic oscillator given by

$$\mathrm{Tr}_y e^{-s(\widehat{p}_y^2 + e^2F^2\widehat{y}^2)} = \frac{1}{2}\frac{1}{\sinh(seF)}. \tag{6.20}$$

We finally obtain

$$I = (VolR^2)\frac{1}{4\pi}\left(\frac{eF}{1 + \frac{\Theta eF}{2}}\right)\frac{1}{\sinh(seF)}. \tag{6.21}$$

Thus the effective action (6.12) is given by

$$\Gamma \propto -\int_0^\infty \frac{ds}{s}\prod_{j=1}^{p}\frac{1}{4\pi}\left(\frac{x_j/2}{1 + \frac{\Theta x_j}{4}}\right)\frac{1}{\sinh(\frac{sx_j}{2})}e^{-sM^2} + \text{ constant}, \tag{6.22}$$

where $x_j = 2eF$.

6.3. Gravitational Anomaly for Spin-$\frac{1}{2}$ Fields

Using Eq. (6.22) concerning the total amplitude of the residual interaction we get

$$\mathcal{Z} = -\int_0^\infty \frac{ds}{s}\prod_{j=1}^{2k+1}\frac{1}{4\pi}\left(\frac{\frac{1}{2}x_j}{\sinh(\frac{sx_j}{2})}\right)\left(\frac{1}{1 + \Theta\frac{x_j}{4}}\right)\exp(-sM^2). \tag{6.23}$$

After s-integration we finally get

$$\mathcal{Z} = -\frac{1}{(4\pi)^{2k+1}}\frac{1}{M^2}\prod_{j=1}^{2k+1}\frac{\frac{1}{2}x_j}{4\pi\sinh(\frac{1}{2}x_j)}\left(\frac{1}{1 + \Theta\frac{x_j}{4}}\right). \tag{6.24}$$

This equation also can be written as

$$I_{\frac{1}{2}} = -i\frac{1}{(2\pi)^{2k+1}}R(\varepsilon^{(i)}, p^{(j)})\widehat{A}_{\Theta}(X), \tag{6.25}$$

where

$$\widehat{A}_{\Theta}(X) = \prod_{j=1}^{2k+1}\left(\frac{\frac{1}{2}x_j}{\sinh(\frac{1}{2}x_j)}\right)\left(\frac{1}{1+\Theta\frac{x_j}{4}}\right), \tag{6.26}$$

is the noncommutative roof-genus. Roof-genus enters in the Atiyah-Singer theorem, thus our computation leads evidently to a noncommutative continuous deformation of the Atiyah-Singer theorem.

6.4. Gravitational Anomaly for Spin-$\frac{3}{2}$ Fields

Now, we would like to compute the one-loop diagram of $2k+2$ external gravitons of momentum $p_\mu^{(i)}$ and polarizations $\varepsilon_{\mu\nu}^{(i)}$ with $i = 1, \ldots, 2k+2$ coupled to Rarita-Schwinger fields of spin $\frac{3}{2}$. In order to make this computation we will use also Adler's prescription to find an equivalent diagram with $2k+1$ external gravitons with only one insertion of an axial factor $\frac{1}{2}(1-\overline{\Gamma})$ and $2k+1$ non-anomalous vertices.

We start from a gauge fixed linearized noncommutative contributions

$$L_1^{RS} = \frac{1}{4}ih^{\alpha\beta} * \psi_\mu * \Gamma_\alpha \overleftrightarrow{\partial} (\frac{1-\overline{\Gamma}}{2})\psi^\mu \tag{6.27}$$

and

$$L_2^{RS} = \frac{1}{2}iG_{\sigma\alpha\nu} * \overline{\psi}^\sigma * \Gamma^\nu\psi^\alpha, \tag{6.28}$$

where $G_{\sigma\alpha\nu} = \big(\partial_\sigma h_{\alpha\nu} - \partial_\alpha h_{\sigma\nu}\big)$. The analisis of gauge fixing involves the existence of ghost fields that modify the total amplitude and it is modified by $I_{\frac{3}{2}}(total) = I_{\frac{3}{2}}(gravitino) - I_{\frac{1}{2}}$.

Using the Feynman rules associated to L_1^{RS} and L_2^{RS} in each one of the $2k+1$ vertices we have for the i-th vertex there is the insertion of a factor: $-\frac{1}{4}i\varepsilon_\mu^{(i)}(p+p')^\mu\frac{1}{p^2-M^2}\exp\big(-\frac{i}{2}\Theta^{\rho\sigma}p_\rho p'_\sigma\big)$, where p is the incoming momentum and p' is the outgoing momentum. The whole contribution is encoded in the amplitude $\widehat{\mathcal{Z}}(\varepsilon^{(i)}, p^{(j)}, \Theta)$. The total amplitude is then given by

$$I_{\frac{3}{2}} = 2^{2k+1}iM^2R(\varepsilon^{(i)}, p^{(j)}) \cdot \widetilde{\mathcal{Z}}(\varepsilon^{(i)}, p^{(j)}, \Theta), \tag{6.29}$$

where $R(\varepsilon^{(i)}p^{(j)})$ is the same kinematical factor (6.3),which depends only on the external momenta and polarization vectors and $\widetilde{\mathcal{Z}}$ can be regarded as the amplitude for the coupling of a charged (complex) noncommutative abelian vector field in a loop coupled to $2k+2$ photons of momenta $p^{(j)}$ and polarization tensors $\varepsilon^{(i)}$ in a noncommutative spacetime. This noncommutative residual interaction is described by the corresponding interaction lagrangians

$$L_1^{(res)} = \frac{1}{4} A^\mu * \overline{\phi}_\sigma * \partial_\mu \phi^\sigma \tag{6.30}$$

and

$$L_2^{(res)} = \frac{1}{2} G_{\mu\nu} * \overline{\phi}_\sigma * \phi^\sigma, \tag{6.31}$$

where $G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{i}{4}[A^\mu, A^\nu]_*$. The first action (6.30) gives precisely the interaction we discussed in the previous subsection and which consist of D complex scalars with charge $\frac{1}{4}$ coupled to $2k+2$ noncommutative 'photons'. The second lagrangian (6.31) corresponds to a noncommutative magnetic moment which has the usual term $\int d^D x \, \overline{\phi}^\mu * (\partial_\mu A_\nu - \partial_\nu A_\mu) * \phi^\nu$ plus an additional term of the form

$$-\frac{i}{4} \int d^{4k+2} x \, \overline{\phi}^\mu * [A_\mu, A_\nu]_* * \phi^\nu, \tag{6.32}$$

which comes from the quadratic term of the definition of $G_{\mu\nu}$. The linear term is exactly the same as the spin-$\frac{1}{2}$ case thus both terms can be gatthered and corresponds with the computation of $\mathcal{Z}$ of the spin-$\frac{1}{2}$ case in the previous subsection. Thus in this case the only difference lies in the interaction term (6.32). We now proceed to compute this term. We use the cyclicity property of the trace in order to remove the $*$-product arising in the noncommutative commutator $[A_\mu, A_\nu]_*$. With this in mind we get

$$\begin{aligned} -\frac{i}{4} \int d^{4k+2} x \, \Big\{ &\Big(\overline{\phi}_\mu e^{\overleftarrow{\partial}_\alpha \Theta^{\alpha\beta} \overrightarrow{\partial}_\beta} A^\mu \Big) \cdot \Big(A^\nu e^{\overleftarrow{\partial}_\alpha \Theta^{\alpha\beta} \overrightarrow{\partial}_\beta} \phi_\nu \Big) \\ &- \Big(\overline{\phi}_\mu e^{\overleftarrow{\partial}_\alpha \Theta^{\alpha\beta} \overrightarrow{\partial}_\beta} A^\nu \Big) \cdot \Big(A^\mu e^{\overleftarrow{\partial}_\alpha \Theta^{\alpha\beta} \overrightarrow{\partial}_\beta} \phi_\nu \Big) \Big\}. \end{aligned} \tag{6.33}$$

Now, using the gauge (6.13) the only term that contributes is that of second order in Θ in equation (6.33). Reordering all terms and integrating by parts all terms cancel identically, which means that the quadratic term does not contributes to the amplitude. Then the remaining lagrangian is given by

$$L=\int d^{4k+2}x\;\overline{\phi}^{\sigma}*\Big\{-\Big[\partial^{\mu}-ie\Big(e^{\overleftarrow{\partial}_{\alpha}\Theta^{\alpha\beta}\overrightarrow{\partial}_{\beta}}A^{\mu}\Big)\Big]$$

$$\times\Big[\partial_{\mu}-ie\Big(A_{\mu}e^{\overleftarrow{\partial}_{\alpha}\Theta^{\alpha\beta}\overrightarrow{\partial}_{\beta}}\Big)\Big]+M^{2}-\frac{i}{2}F_{\mu\nu}\Big\}*\phi_{\sigma}. \tag{6.34}$$

Then, the effective action reads

$$\widetilde{\mathcal{Z}}=-\mathrm{Tr}\int_{0}^{\infty}\frac{ds}{s}$$

$$\times\Big\{e^{-s\Big[-\Big(\partial^{\mu}-ie\big(e^{\overleftarrow{\partial}_{\alpha}\Theta^{\alpha\beta}\overrightarrow{\partial}_{\beta}}A^{\mu}\big)\Big)\Big(\partial_{\mu}-ie\big(A_{\mu}e^{\overleftarrow{\partial}_{\alpha}\Theta^{\alpha\beta}\overrightarrow{\partial}_{\beta}}\big)\Big)+M^{2}-\frac{i}{2}F_{\mu\nu}\Big]}-e^{-s}\Big\}. \tag{6.35}$$

All exponential terms factorize and we can see that the problem reduces to the computation for spin-$\frac{1}{2}$ from the previous subsection plus the contribution of the factor $\mathrm{tr}\exp\big(-\frac{1}{2}sF^{\mu\nu}_{(j)}\big)=2\cosh(sx_{j})$.

Thus we get the amplitude $\widetilde{Z}$ by considering the ghost contribution and it yields

$$\widetilde{\mathcal{Z}}=-\int_{0}^{\infty}\frac{ds}{s}\prod_{j=1}^{2k+1}\frac{1}{4\pi}\bigg(\frac{\frac{1}{2}x_{j}}{\sinh(\frac{sx_{j}}{2})}\bigg)\bigg(\frac{1}{1+\Theta\frac{x_{j}}{4}}\bigg)\bigg(-1+\sum_{i=0}^{2k+1}2\cosh(x_{i})\bigg)\exp(-sM^{2}). \tag{6.36}$$

After integrating out the s-variable we finally get

$$\widetilde{\mathcal{Z}}=-\frac{1}{(4\pi)^{2k+1}}\frac{1}{M^{2}}\prod_{j=1}^{2k+1}\frac{\frac{1}{2}x_{j}}{4\pi\sinh(\frac{1}{2}x_{j})}\bigg(\frac{1}{1+\Theta\frac{x_{j}}{4}}\bigg)\bigg(-1+\sum_{i=0}^{2k+1}2\cosh(x_{i})\bigg). \tag{6.37}$$

Then, the total amplitude for Rarita-Schwinger fields is given by

$$I_{\frac{3}{2}}(total)=-i\frac{1}{(2\pi)^{2k+1}}R(\varepsilon^{(i)},p^{(j)})\prod_{j=1}^{2k+1}\frac{\frac{1}{2}x_{j}}{4\pi\sinh(\frac{1}{2}x_{j})}\bigg(\frac{1}{1+\Theta\frac{x_{j}}{4}}\bigg)\bigg(-1+\sum_{i=0}^{2k+1}2\cosh(x_{i})\bigg). \tag{6.38}$$

Then, the total amplitude for Rarita-Schwinger fields is also modified by the same Θ-dependent factor justly as the spin-$\frac{1}{2}$ fields of the previous section.

7. Noncommutative Mixed Anomalies

7.1. Mixed Anomaly for Spin $\frac{1}{2}$-Fields

In this subsection we will compute mixed anomalies which include not only the coupling of chiral fermions to gravity but also to nonabelian gauge fields. Noncommutative gauge anomalies, for the case of Yang-Mills fields have been computed in a number of papers, see for instance [9,11,12,14,15] for planar diagrams with gauge group $U(N)$.

We will consider a noncommutative spacetime of even dimension $D = 2n$ and we compute one-loop amplitudes of r external gluons and $n+1-r$ external gravitons. We will concentrate in anomalous diagrams with $n+1-r$ =even. Recent results concerning the computation of chiral gauge anomalies in Yang-Mills theories in an even number of spacetime dimensions was performed through the Wess-Zumino consistency condition in the reference [12]. In the present paper we apply the procedure of [11]. For the case of non-planar diagrams there has been some previous work in [10,26,27]. The analysis can be extended to other gauge groups by introducing the Seiberg-Witten map as in references [20,22,23,24,25].

Before the evaluation of the relevant diagrams let us review briefly some ideas of noncommutative Yang-Mills theory. Consider a gauge theory with a hermitian connection, invariant under a symmetry Lie group $\mathbf{G}$, with gauge fields A_μ and gauge transformations: $\delta_\lambda A_\mu = \partial_\mu \lambda + i\big[\lambda, A_\mu\big]$, with $\lambda = \lambda^i T_i$, and T_i are the generators of the Lie algebra $\mathcal{G}$ of the group $\mathbf{G}$, in the adjoint representation. In the noncommutative Yang-Mills theory, the product of functions on the spacetime manifold is promoted to the Moyal product. The above transformations are generalized for the noncommutative theory as, $\delta_\lambda \widehat{A}_\mu = \partial_\mu \widehat{\Lambda} + i\big[\widehat{\Lambda}, \widehat{A}_\mu\big]_*$, where the commutators are defined as $\big[A, B\big]_* \equiv A*B - B*A$. Due to noncommutativity, a generic commutator takes values in the universal enveloping algebra $\mathcal{U}(\mathcal{G}, \mathbf{R})$ of the Lie algebra $\mathcal{G}$ in the representation $\mathbf{R}$ (for more details see, for instance [50]). In particular, $\big[\widehat{\Lambda}, \widehat{A}_\mu\big]_*$ take values in the universal enveloping algebra $\mathcal{U}(su(N), \mathbf{ad})$ of the Lie algebra $su(N)$ (where, for instance, $\mathbf{G} = SU(N)$) in the adjoint representation $\mathbf{ad}$. Therefore, $\widehat{\Lambda}$ and the gauge fields $\widehat{A}_\mu$ will also take values in this algebra. Let us write for instance $\widehat{\Lambda} = \widehat{\Lambda}^I T_I$ and $\widehat{A} = \widehat{A}^I T_I$, then,

$$\big[\widehat{\Lambda}, \widehat{A}_\mu\big]_* = \big\{\widehat{\Lambda}^I, \widehat{A}^J_\mu\big\}_* \big[T_I, T_J\big] + \big[\widehat{\Lambda}^I, \widehat{A}^J_\mu\big]_* \big\{T_I, T_J\big\}, \tag{7.1}$$

where $\{A, B\}_* \equiv A*B + B*A$ is the noncommutative anticommutator and the indices I, J, K etc, run over the number of generators of the enveloping algebra. Thus all the

products of the generators T_I will be needed in order to close the algebra $\mathcal{U}(\mathcal{G}, \mathbf{ad})$. Its structure can be obtained by successive computation of commutators and anticommutators starting from the generators of $\mathcal{G}$, until it closes,

$$\big[T_I, T_J\big] = i f_{IJ}{}^K T_K, \qquad \big\{T_I, T_J\big\} = d_{IJ}{}^K T_K. \tag{7.2}$$

The field strength is defined as $\widehat{F}_{\mu\nu} = \partial_\mu \widehat{A}_\nu - \partial_\nu \widehat{A}_\mu - i[\widehat{A}_\mu, \widehat{A}_\nu]_*$, hence it takes also values in $\mathcal{U}(\mathcal{G}, \mathrm{ad})$.

Thus axial anomalies in $2n$ dimensions can be obtained by computing the amplitude associated to the one-loop diagram with r external gluons and $n+1-r$ external gravitons. In the noncommutative case at each gluon vertex we have to insert

$$-i\Gamma^\mu T_L^I \exp\big(-\frac{i}{2}\Theta^{\rho\sigma} p_{1\rho} p_{2\sigma}\big)\delta(p_1 + p_2 + k), \tag{7.3}$$

where T_L^I is the generator of the enveloping algebra $\mathcal{U}(\mathcal{G}, \mathbf{R})$ in the representation $\mathbf{R}$ furnished by the left-handed fermions. The group theory factor associated with a given diagram is: $\mathrm{Tr}\big(T_L^{I_1} \cdot T_L^{I_2} \dots T_L^{I_r}\big)$. After this factor is extracted, in each gluon vertex we have a factor given by

$$-i\Gamma^\mu \exp\big(-\frac{i}{2}\Theta^{\rho\sigma} p_{1\rho} p_{2\sigma}\big)\delta(p_1 + p_2 + k). \tag{7.4}$$

On the other hand in each graviton vertex we have to insert the factor

$$-\frac{i}{4}\varepsilon^{\mu\nu}\Gamma_\mu\left(\frac{1-\overline{\Gamma}}{2}\right)(p+p')_\nu \exp\left(-\frac{i}{2}\Theta^{\mu\nu} p_\mu p'_\nu\right).$$

Dirac algebra of matrices Γ can be carried out and then trace does not distinguish of the graviton and gluon vertices. Thus the kinematic factor $R(\varepsilon^{(i)}, p^{(j)})$ is exactly the same. After that, graviton vertex correspond to a massive complex scalar fields of charge $\frac{1}{4}$ interacting with "photons" which give rise to a noncommutative effective theory of charged scalars coupled to external photons of the same type as that described in the previous section. After Dirac and group trace for the gluon vertex (7.4) we have $-i\exp\big(-\frac{i}{2}\Theta^{\rho\sigma} p_{1\rho} p_{2\sigma}\big)$. This remaining noncommutative vertex corresponds to a coupling of a scalar fields to the mentioned massive complex scalars. Thus the remaining effective diagram consist of external scalar and photon fields coupled to complex scalar fields, with the usual propagators $i/(p^2 - M^2)$, obeying noncommutative interactions.

Similarly to the commutative case now we have to restrict trace formula to the symmetric trace since noncommutativity respects the symmetry under permutations of external lines [51]. Thus the factor $\mathcal{Z}'$ is given by

$$\mathcal{Z}'(\Theta) = -\mathrm{STr}\Big[T_L^{I_1} T_L^{I_2} \dots T_L^{I_r} \exp\Big(-\frac{i}{2}\Theta^{\rho\sigma} \sum_{\ell=1}^{r-1} p_{1\rho}^{\ell} p_{2\sigma}^{\ell}\Big)\Big]$$
$$\times \Big(\frac{\partial}{\partial M^2}\Big)^r \int \frac{ds}{s} \prod_{j=1}^{2k+1} \Bigg[\frac{\frac{1}{2}x_j}{4\pi \sinh(\frac{s x_j}{2})} \frac{1}{\big(1+\Theta\frac{x_j}{4}\big)}\Bigg] \exp(-sM^2), \tag{7.5}$$

where the derivative stands, as in the commutative case, that the $-i\cdot\exp\big(-\frac{i}{2}\Theta^{\rho\sigma}p_{1\rho}p_{2\sigma}\big)$ vertex has can be obtained through a derivative with respect the mass square M^2, *i.e.* $\frac{i}{p^2-M^2}(-i)\frac{i}{p^2-M^2} = \frac{\partial}{\partial M^2}\big[\frac{i}{p^2-M^2}\big]$. Here STr is the symmetrized trace in the factor corresponding the gauge amplitude is constructed by insering in each vertex a factor: $-i\Gamma^{\mu}\exp\big(-\frac{i}{2}\Theta^{\rho\sigma}\ell_{\rho}^{(j)}\ell_{\sigma}^{(j+1)}\big)$. Then the symmetrized trace is given by

$$\mathrm{Tr}\big[T_L^{I_1}T_L^{I_2}\dots T_L^{I_r}\big]\Big\{\cos\frac{\ell^{(1)}\Theta\ell^{(2)}}{2}\cdot\cos\frac{\ell^{(3)}\Theta\ell^{(4)}}{2}\dots\cos\frac{\ell^{(r-1)}\Theta\ell^{(r)}}{2} + \text{all permutations}\Big\},$$

where $\ell^{(i)}\Theta\ell^{(i+1)} \equiv \Theta^{\rho\sigma}\ell_{\rho}^{(i)}\ell_{\sigma}^{(i+1)}$.

For instance for $r=4$ we have

$$\mathrm{Tr}\big[T_L^{I_1}T_L^{I_2}T_L^{I_3}T_L^{I_4}\big]\Big\{\cos\frac{\ell^{(1)}\Theta\ell^{(2)}}{2}\cdot\cos\frac{\ell^{(3)}\Theta\ell^{(4)}}{2} + \cos\frac{\ell^{(1)}\Theta\ell^{(3)}}{2}\cdot\cos\frac{\ell^{(2)}\Theta\ell^{(4)}}{2}$$
$$+\cos\frac{\ell^{(1)}\Theta\ell^{(4)}}{2}\cdot\cos\frac{\ell^{(2)}\Theta\ell^{(3)}}{2}\Big\}.$$

After s-integration we finally get that the total mixing anomaly is given by

$$I'_{\frac{1}{2}} = -\mathrm{STr}\Big[T_L^{I_1}T_L^{I_2}\dots T_L^{I_r}\exp\big(-\frac{i}{2}\Theta^{\rho\sigma}\sum_{\ell=1}^{r-1}p_{1\rho}^{\ell}p_{2\sigma}^{\ell}\big)\Big]$$
$$\times\frac{i}{(2\pi)^{\frac{n}{2}}}R(\varepsilon^{(i)},p^{(j)})\prod_{j=1}^{\frac{n}{2}}\frac{\frac{1}{2}x_j}{4\pi\sinh(\frac{1}{2}x_j)}\frac{1}{\big(1+\Theta\frac{x_j}{4}\big)}. \tag{7.6}$$

The interpretation of the gauge anomaly is justy as in case of chiral gauge anomaly in Yang-Mills theory. In the case of $U(N)$ gauge group, as was described in Ref. [12], the noncommutativity imposes more restrictive conditions for anomaly cancelation. Thus in

order a noncommutative gauge theory be anomaly free this theory must be non-chiral. In four dimensions noncommutative chiral gauge field theories with $U(N)$ group with adjoint matter is anomaly free but is not loger true in higher dimensions. For instance in our present case of $D = 4k+2$ dimensions it has been showed [12] that in for adjoint matter, chiral anomaly is non-vanishing and it is precisely the $2N$ times the anomaly in the fundamental representation.

7.2. Mixed Anomaly for Spin-$\frac{3}{2}$ Fields

Similarly to the case of noncommutative mixed anomalies of gauge and gravitational fields coupled to complex chiral spin-$\frac{1}{2}$ we we can compute the mixed anomalies for spin-$\frac{3}{2}$ case. Then we have

$$\widetilde{\mathcal{Z}}'(\Theta) = -\mathrm{STr}\Big[\widetilde{T}_L^{I_1}\widetilde{T}_L^{I_2}\ldots\widetilde{T}_L^{I_r}\exp\big(-\frac{i}{2}\Theta^{\rho\sigma}\sum_{\ell=1}^{r}p^{\ell}_{1\rho}p^{\ell}_{2\sigma}\big)\Big]$$
$$\times\Big(\frac{\partial}{\partial M^2}\Big)^r\int\frac{ds}{s}\prod_{j=1}^{2k+1}\Big[\frac{\frac{1}{2}x_j}{4\pi\sinh(\frac{sx_j}{2})}\frac{1}{\big(1+\Theta\frac{x_j}{4}\big)}\Big]\Big(-1+\sum_{i=0}^{2k+1}2\cosh(x_i)\Big)\exp(-sM^2). \tag{7.7}$$

After s-integration we finally get

$$I'_{\frac{3}{2}} = -\mathrm{STr}\Big[\widetilde{T}_L^{I_1}\widetilde{T}_L^{I_2}\ldots\widetilde{T}_L^{I_r}\exp\big(-\frac{i}{2}\Theta^{\rho\sigma}\sum_{\ell=1}^{r}p^{\ell}_{1\rho}p^{\ell}_{2\sigma}\big)\Big]$$
$$\times\frac{i}{(2\pi)^{\frac{n}{2}}}R(\varepsilon^{(i)},p^{(j)})\prod_{j=1}^{\frac{n}{2}}\frac{\frac{1}{2}x_j}{4\pi\sinh(\frac{1}{2}x_j)}\frac{1}{\big(1+\Theta\frac{x_j}{4}\big)}\Big(\sum_{i=1}^{\frac{n}{2}}2\cosh(x_i)\Big), \tag{7.8}$$

where STr stands for the symmetrized trace which is defined as in the previous subsection.

8. Final Remarks

In this paper we have studied axial and chiral gravitational anomalies in the context of noncommutative field theory. An interesting feature is that they are natural higher-dimensional generalizations of the studies of axial and gauge anomalies in noncommutative gauge theories. In order to compute the noncommutative effects we have used a linearization of a noncommutative deformation of Einstein theory [32], but in principle, we could used any other noncommutative theory of gravity. This noncommutative deformation of

linear gravity has been coupled to noncommutative chiral fermions and we have assumed that gravity as well as matter are deformed with the same deformation parameter Θ. Thus, we focus on the interaction action of chiral fermions and the gravitational field. We have provided the Feynman rules of this noncommutative theory, in particular (3.6) was the necessary rule to determine the anomalies of planar diagrams. Anomalies coming from non-planar diagrams were not considered in the present paper. The only modification appears on the vertices of Feynman diagrams and we use them to compute a series of processes involving gravitational anomalies.

After discussing the Feynman rules we have computed the noncommutative contribution to the gravitational axial (ABJ) anomaly leading to the pion decay into two photons. This noncommutative extension of the Delbourgo-Salam anomaly is obtained by using the dimensional regularization method and we found that it gives precisely a noncommutative deformation of signature $\widehat{\tau}(X)$ which is the spacetime analogous to the group signature worked out in Ref. [35].

As in the commutative case, noncommutative Delbourgo-Salam anomaly does not spoil diffeomorphism or local Lorentz gauge invariance at the quantum level. However noncommutativity might affect also these gauge symmetries as far as Lorentz transformations and diffeomorphism symmetries are affected in noncommutative field theories.

In the two-dimensional case of the pure gravitational chiral anomaly, we have computed diffeomorphism anomaly and we found that the noncommutativity does not affect the effective action $\Gamma(Q)$ and therefore the anomaly is the same than in the usual commutative case obtained in [40]. This is also done in the general case of $D = 4k+2$ dimensions. The anomaly was obtained by finding first a noncommutative residual interaction of a complex scalar field with an $U(1)$ gauge field. Here as usual in the commutative case, for each coupling vertex we have translated the graviton and chiral fermion interaction to the problem to the problem of the vertex of a complex scalar field coupled to external non-dynamical external photons. The effective action is computed by using a two-dimensional noncommutative version Schwinger model. We find a noncommutative deformation of the effective action given by the expression (6.25) and (6.26). The computation of the anomaly for a loop of spin-$\frac{3}{2}$ fields was performed and it was obtained also a noncommutative correction given by the expression (6.38).

Mixed anomalies also were computed in within this context and there is also a noncommutative modification given by (7.6) and (7.8) for spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ fields respectively.

There are several interesting points concerning the results of this work. One of them concerns the application to the different ten-dimensional supergravities coming from string theory. It would be interesting to compute the gauge and gravitational anomalies due to anti-symmetric p-form fields in a noncommutative background and to look for the conditions for the cancellation of these noncommutative anomalies in type I and type II supergravities ten dimensions. Before of solving these problem perhaps one first would address the problem of to give a sensible theory of noncommutative extension of gauge theory for higher rank form potentials in higher dimensions.

Another interesting problem is the computation of gravitational anomalies due to non-planar diagrams following [10,26,27]. In the present paper we limited to compute chiral gauge anomalies for $U(N)$ group. We would like to extend the computation to other gauge groups by using the Seiberg-Witten map following some literature on this subject [20,22,23,24,25]. We would like to apply the Seiberg-Witten map for the gravitational sector as was discussed in [29,30,35,36].

One of the most interesting problems in to connect our results given by Eqs. (6.25), (6.26)and (6.38) with the Atiyah-Singer index theorem for families of elliptic operators and to give explicit formulas for these noncommutative anomalies in terms of the invariant polynomials describing Pontrjagin and Chern characteristic classes. This is left for a future communication. A description in terms of the Wess-Zumino consistency condition similarly to [12] it worth to provide for the case of gravity. In order to do that the results of Ref. [52] would be important.

Finally, it would be very interesting also to pursue a suitable global approach, including gravitational global anomalies [42], and compare it with the results given recently by Perrot [53] in the computation of noncommutative gravitational anomalies using different global tools.

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